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BOUNDEDNESS FOR PERTURBED DIFFERENTIAL EQUATIONS VIA LYAPUNOV EXPONENTS

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Abstract. In this paper we investigate the stability of solutions of the perturbed differential equations with the positive order of the perturbation by using the notion of the Lyapunov exponent of unperturbed equations and an integral inequality of Bihari type.

1. Introduction

The notion of the Lyapunov exponents which originated in the pioneering work of Lyapunov [9] plays an important role in the stability theory of the dynamical systems. In particular, the Lyapunov exponents provide precise information about the asymptotic behavior of solutions of nonautonomous ordinary differential equations.

Barreira and Pesin [2] proved the Lyapunov Stability Theorem with the order $q > 1$ of the perturbation by using the notion of Lyapunov regularity. Furthermore, Barreira and Silva [3] generalized the concept of Lyapunov exponent to transformations that are not necessarily differentiable. They also discussed the relation of the new Lyapunov exponents with the dimension theory of dynamical systems for invariant sets of continuous transformations. Choi et al.[6, 7] studied the stability for linear dynamic equations on time scales and nonlinear difference equations by using the Lyapunov-type functions and the notions of similarity, respectively.

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In this paper we investigate the stability of solutions of the perturbed differential equations with the positive order of the perturbation by using the notion of the Lyapunov exponent of unperturbed equations and an integral inequality of Bihari type.

2. Preliminaries

We recall some results about Lyapunov exponents for differential equations presented in [1, 2, 5, 10].

We consider a linear differential equation

$$v' = A(t)v, \tag{2.1}$$

where $A(t) \in C(\mathbb{R}, \mathbb{C}^n)$ and

$$\sup\{|A(t)| : t \in \mathbb{R}\} < \infty. \tag{2.2}$$

Then it follows that for every $v_0 \in \mathbb{C}^n$ there exists a unique solution $v(t) = v(t, 0, v_0)$ of (2.1) that is defined for every $t \in \mathbb{R}$ with $v(0) = v_0$.

Remark 2.1. If $A(t) = A$ for all $t \geq 0$ in (2.1), then the trivial solution $v = 0$ of (2.1) is exponentially stable if and only if the real part of every eigenvalue of the matrix A is negative.

By means of Lyapunov's estimate it has been proved that the Lyapunov exponents of linear differential equations are finite in the case when the coefficients are bounded.

Lemma 2.2. [1, Lyapunov's estimate] *For any solution $v(t)$ of (2.1), the inequality*

$$|v(t_0)|e^{-\int_{t_0}^t |A(s)|ds} \leq |v(t)| \leq |v(t_0)|e^{\int_{t_0}^t |A(s)|ds},\ t \geq t_0 \geq 0,$$

is valid.

In order to characterize the stability of the trivial solution in general case we introduce the notion of Lyapunov exponent due to A. M. Lyapunov [9].

Definition 2.3. [2] We define the *Lyapunov exponent* $\chi^+ : \mathbb{C}^n \to \mathbb{R} \cup \{-\infty\}$ of Equation (2.1) by the formula

$$\chi^+(v) = \lim_{t\to\infty} \sup \frac{1}{t} \log |v(t)|,\ v \in \mathbb{C}^n,$$

where $v(t)$ is the unique solution of (2.1) satisfying the initial condition $v(0) = v$.

Example 2.4. [10] *The following linear scalar differential equation*

$$v'(t) = (\cos(\ln|t|) + \sin(\ln|t|))v \tag{2.3}$$

has the general solution $v(t) = \exp(t\sin(\ln|t|))v_0$. *Thus the Lyapunov spectrum defined by*

$$\{\lambda = \lim_{j\to\infty} \frac{1}{t_j} \ln|v(t_j, v)| \text{ for some sequence } t_j \to \infty \text{ as } j \to \infty\}$$

becomes

$$\{\lim_{j\to\infty} \frac{t_j}{t_j} \sin(\ln|t_j|)\} = [-1, 1].$$

Thus we obtain the Lyapunov exponent of (2.3) given by

$$\chi^+(v) = \lim_{t\to\infty} \sup \frac{1}{t} \ln|v(t)| = 1, v \neq 0.$$

Remark 2.5. [2, 10] The Lyapunov exponent χ^+ satisfies the following properties:

(i) $\chi^+(\alpha v) = \chi^+(v)$ for each $v \in \mathbb{C}^n$ and $\alpha \neq 0$;
(ii) $\chi^+(v + w) \leq \max\{\chi^+(v), \chi^+(w)\}$ for each $v, w \in \mathbb{C}^n$;
(iii) $\chi^+(0) = -\infty$;
(iv) χ^+ is independent of the choice of norm on $\mathbb{C}^n$.

Theorem 2.6. [1, Theorem 2.3.1] *If* $\sup\{|A(t)| : t \in \mathbb{R}^+\} \leq M$, *then any nontrivial solution* $v(t) = v(t, 0, v)$ *of Equation (2.1) has a finite Lyapunov exponent, and* $-M \leq \chi^+(v) \leq M$.

Remark 2.7. [2, Theorem 1.2.1] The function χ^+ attains only finitely many distinct values $\chi_1^+ = \chi^+(v_1) < \cdots < \chi_s^+ = \chi^+(v_s)$ on $\mathbb{C}^n \setminus \{0\}$ where $s \leq n$. Each number χ^+ occurs with some multiplicity k_i so that $\sum_{i=1}^s k_i = n$.

3. Main results

In this section we investigate the boundedness of solutions of nonlinear differential equations by using the notion of Lyapunov exponents and an integral inequality of Bihari type.

We give in detail the proof of a well known result about the stability of solutions of (2.1) by using the notion of Lyapunov exponents.

Lemma 3.1. [2] *Suppose that there are finite distinct values* $\chi_1^+ < \cdots < \chi_s^+$ *on* $\mathbb{C}^n \setminus \{0\}$ *where* $s \leq n$.

(i) *For every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that we have*

$$|v(t)| \leq C_\varepsilon e^{(\chi_s^+ + \varepsilon)t}|v(0)|, \ t \geq 0, \tag{3.1}$$

where $v(t)$ is any solution of (2.1).

(ii) *If $\chi_s^+ < 0$, then the trivial solution $v(t) = 0$ of (2.1) is exponentially stable.*

Proof. Let $v_0 \in \mathbb{C}^n \setminus \{0\}$ and $\varepsilon > 0$ be given. Then we have

$$\begin{aligned} \chi_s^+(v_0) &= \chi_s^+(\frac{v_0}{|v_0|}), \\ \frac{\log|v(t,0,v_0)|}{t} &< \chi_s^+ + \frac{\varepsilon}{2}, \ t \geq 0. \end{aligned}$$

Thus we obtain

$$\begin{aligned} |v(t,0,\frac{v_0}{|v_0|})| &< e^{(\chi_s^+ + \frac{\varepsilon}{2})t} \\ &= e^{-\frac{\varepsilon}{2}t}e^{(\chi_s^+ + \varepsilon)t} \\ &\leq C_\epsilon e^{(\chi_s^+ + \varepsilon)t}, \ t \geq 0, \end{aligned}$$

since $e^{-\frac{\varepsilon}{2}t}$ is bounded for each $t \geq 0$. Hence we have

$$|v(t,0,v_0)| \leq C_\epsilon e^{(\chi_s^+ + \varepsilon)t}|v_0|, \ t \geq 0.$$

Next, we can choose $\varepsilon > 0$ sufficiently small so that $\chi_s^+ < -\varepsilon < 0$. Hence we obtain

$$\begin{aligned} |v(t,0,v_0)| &\leq C_\epsilon e^{(\chi_s^+ + \varepsilon)t}|v_0| \\ &\leq C_\epsilon e^{-\alpha t}|v_0|, \ t \geq 0, \end{aligned}$$

where α is a positive constant such that $\chi_s^+ + \varepsilon < -\alpha < 0$. Hence the trivial solution $v(t) = 0$ of (2.1) is exponentially stable. This completes the proof. □

REMARK 3.2. When $A(t) = A$, we note that the real parts of eigenvalues of A does not need to coincidence to Lyapunov exponents of the system.

We give an example to illustrate Remark 3.2.

EXAMPLE 3.3. [1] *We consider the differential system*

$$x' = \mathrm{daig}[1, 2, 3]x$$

and its two fundamental systems of solutions

$$X_1(t) = \left\{ \begin{bmatrix} e^t \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ e^{2t} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ e^{3t} \end{bmatrix} \right\},$$

$$X_2(t) \quad = \quad \left\{ \begin{bmatrix} e^t \\ 0 \\ e^{3t} \end{bmatrix}, \begin{bmatrix} 0 \\ e^{2t} \\ e^{3t} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ e^{3t} \end{bmatrix} \right\}.$$

Then we have

$$\begin{aligned} \chi^+(X_1(t)) &= \{1,2,3\} = \mathrm{sp}(A), \\ \chi^+(X_2(t)) &= \{3\} \neq \{1,2,3\} = \mathrm{sp}(A). \end{aligned}$$

We now consider a nonlinear differential equation as a perturbation of (2.1)

$$u' = A(t)u + f(t,u), \tag{3.2}$$

where $f(t,0) = 0$ and $f : \mathbb{R}^+ \times H(0) \to \mathbb{C}^n$ is continuous and for every $u, v \in H(0)$ and $t \geq 0$, we have

$$|f(t,u) - f(t,v)| \leq K|u - v|^q, \tag{3.3}$$

where K and q are positive constants. Here $H(0)$ is a neighborhood of 0 in $\mathbb{C}^n$. Let $u(t) = u(t,0,u_0)$ be a unique solution of (3.2) with the initial condition $u(0) = u_0$ defined on $\mathbb{R}^+$. See [2, 8] for the existence of solutions.

Let $\chi_{\max}$ be the maximal value of the Lyapunov exponent of (2.1).

Theorem 3.4. [2, Theorem 1.4.3] *Assume that*

(i) $q > 1$ *in (3.3) and* $\chi_{\max} < 0$*;*

(ii) $(q-1)\chi_{\max} + \gamma < 0$, *where* γ *is the regular coefficient.*

Then the trivial solution $u(t) = 0$ *of the perturbed equation (3.2) is exponentially stable.*

Theorem 3.5. *Assume that*

(i) $\chi_{\max} < 0$*;*

(ii) *the perturbation* f *satisfies the following:*

$$|f(t,u) - f(t,v)| \leq K|u - v|, \ K > 0, \tag{3.4}$$

(iii) $\chi^+_{\max} + \varepsilon + C_\varepsilon K < 0$ *for sufficiently small* $\varepsilon > 0$ *and a constant* $C_\varepsilon > 0$.

Then the trivial solution $u(t) = 0$ *of (3.2) is exponentially stable.*

Proof. It follows from Lemma 3.1 that for sufficiently small $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that we have

$$\begin{aligned} |V(t,s)| &\leq C_\varepsilon e^{(\chi^+_{\max}+\varepsilon)(t-s)} \\ &\leq C_\varepsilon e^{-\alpha(t-s)}, \ t \geq 0, \end{aligned} \tag{3.5}$$

where $V(t,s)$ is the Cauchy matrix of (2.1) and $\alpha = -\chi^+_{\max} - \varepsilon$. From (3.5) and (3.4), we have

$$\begin{aligned}|u(t)| &\leq |V(t,0)||u_0| + \int_0^t |V(t,s)||f(s,u(s))|ds \\ &\leq C_\varepsilon e^{-\alpha t}|u_0| + \int_0^t C_\varepsilon e^{-\alpha(t-s)}K|u(s)|ds,\ t \geq 0,\end{aligned}$$

where $u(t)$ is any solution of (3.2). Putting $w(t) = e^{\alpha t}|u(t)|$, it follows from Gronwall inequality that we have

$$\begin{aligned}w(t) &\leq C_\varepsilon|u_0| + \int_0^t C_\varepsilon K w(s)ds \\ &\leq C_\varepsilon|u_0|e^{C_\varepsilon K t},\ t \geq 0.\end{aligned}$$

Thus we obtain

$$\begin{aligned}|u(t)| &\leq C_\varepsilon|u_0|e^{-(\alpha - C_\varepsilon K)t} \\ &= C_\varepsilon|u_0|e^{-\beta t},\ t \geq 0,\end{aligned}$$

where $-\beta = \chi^+_{\max} + \varepsilon + C_\varepsilon K < 0$. Hence $v = 0$ of (3.2) is exponentially stable. The proof is complete. □

We need the following integral inequality of Bihari type to prove our main result.

Lemma 3.6. [4, Theorem 4.9] *Let $y, a \in C(\mathbb{R}_+)$, and $\alpha > 0$ be a constant with $\alpha \neq 1$. Then*

$$y(t) \leq y_0 + \int_{t_0}^t a(s)y^\alpha(s)ds \tag{3.6}$$

for $t \in t_0$ implies

$$y(t) \leq [y_0^{1-\alpha} + (1-\alpha)\int_{t_0}^t a(s)ds]^{\frac{1}{1-\alpha}}, t \in [t_0, T), \tag{3.7}$$

where T is given by

$$T = \sup\{t \in [t_0, \infty)\ :\ y_0^{1-\alpha} + (1-\alpha)\int_{t_0}^t a(s)ds > 0\}. \tag{3.8}$$

We can improve the result of Theorem 3.4 when $0 < q < 1$.

Theorem 3.7. *Assume that $0 < q < 1$ in (3.3) and $\chi_{\max} < 0$. Then every solution $u(t)$ of (3.2) is bounded, i.e., we have an upper bound for every solution $u(t)$ of (3.2)*

$$|u(t)| \leq M(\frac{K}{\alpha})^{\frac{1}{1-q}},\ t \geq 0,$$

where M is a positive constant.

Proof. In view of the proof of Theorem 3.5, we have

$$\begin{aligned} |u(t)| &\leq |V(t,0)||u_0| + \int_0^t |V(t,s)||f(s,u(s))|ds \\ &\leq C_\varepsilon e^{(\chi^+_{\max}+\varepsilon)t} + \int_0^t C_\varepsilon e^{(\chi^+_{\max}+\varepsilon)(t-s)} K|u(s)|^q ds \\ &\leq C_\varepsilon e^{-\alpha t}|u_0| + \int_0^t C_\varepsilon e^{-\alpha(t-s)} K|u(s)|^q ds,\ t \geq 0, \end{aligned}$$

where $u(t)$ is any solution of (3.2) and $\alpha = -\chi^+_{\max} - \varepsilon > 0$. Putting $w(t) = e^{\alpha t}|u(t)|$, we have

$$w(t) \leq C_\varepsilon|u_0| + \int_0^t C_\varepsilon K e^{(1-q)\alpha s} w^q(s) ds,\ t \geq 0,$$

and from Lemma 3.6 we have

$$w(t) \leq [(C_\varepsilon|u_0|)^{1-q} + (1-q)\int_0^t C_\varepsilon K e^{(1-q)\alpha s} ds]^{\frac{1}{1-q}},\ 0 \leq t < T,$$

where T is given by

$$T = \sup\{t \in \mathbb{R}^+ \mid (C_\varepsilon|u_0|)^{1-q} + (1-q)\int_0^t C_\varepsilon K e^{(1-q)\alpha s} ds \geq 0\} = +\infty.$$

Thus we obtain

$$\begin{aligned} |u(t)| &\leq e^{-\alpha t}[(C_\varepsilon|u_0|)^{1-q} + (1-q)\int_0^t C_\varepsilon K e^{(1-q)\alpha s} ds]^{\frac{1}{1-q}} \\ &= [(C_\varepsilon|u_0|)^{1-q} e^{-(1-q)\alpha t} + \frac{C_\varepsilon K}{\alpha}(1 - e^{-(1-q)\alpha t})]^{\frac{1}{1-q}} \\ \leq M & \ (\frac{C_\varepsilon K}{\alpha})^{\frac{1}{1-q}},\ t \geq 0, \end{aligned}$$

where M is a positive constant. Hence every solution $u(t)$ of (3.2) is bounded. The proof is complete. □

REMARK 3.8. We can obtain the boundedness and exponential stability for the perturbed equation (3.2) with the positive order of the perturbation by using the regularity of Lyapunov exponent and an integral inequality of Bihari type.

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